Analysis of Boolean Functions

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"Analysis of Boolean Functions", Ryan O'Donnell

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Boolean Functions

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Applications of Boolean functions:

- Circuit design.
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- Circuit design.
- Learning theory.
- Voting rule for election with n voters and 2 candidates $\{-1, 1\}$; social choice theory.

Majority, Linear Threshold Functions

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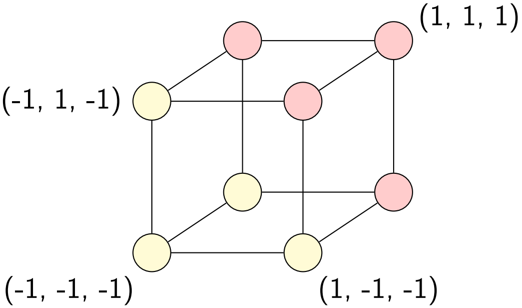
- Majority function: $\mathrm{Maj}_n(x) = \mathrm{sgn}(x_1 + x_2 + \cdots + x_n)$.



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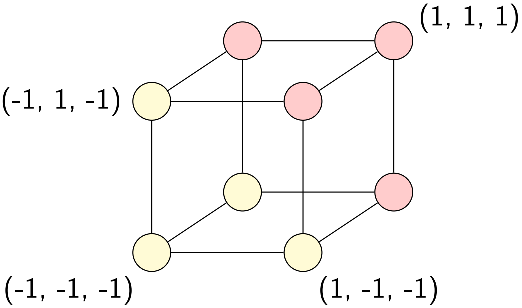

- f is linear threshold function (weighted majority) if

$$f(x) = \mathrm{sgn}(a_0 + a_1x_1 + \cdots + a_nx_n).$$

AND, OR, Tribes

- $-1 \leftrightarrow$ `True`, $1 \leftrightarrow$ `False`.
- $\mathrm{AND}_n(x) = x_1 \wedge x_2 \wedge \cdots \wedge x_n$.
- $\mathrm{OR}_n(x) = x_1 \vee x_2 \vee \cdots \vee x_n$.

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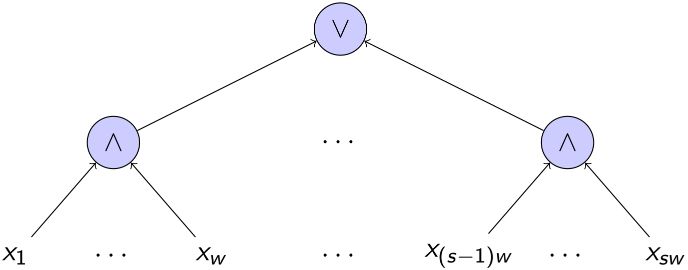
- $\text{OR}_n(x) = x_1 \vee x_2 \vee \cdots \vee x_n$.
- $\text{Tribes}_{w,s}(x_1, \ldots, x_{sw}) = (x_1 \wedge \cdots \wedge x_w) \vee \cdots \vee (x_{(s-1)w} \wedge \cdots \wedge x_{sw})$.
  - $n = ws$ is number of voters.
  - s tribes, w people per tribe.



Influence

Definition

Impartial culture assumption: n votes independent, uniformly random: $\mathbf{x} \sim \{-1, 1\}^n$.

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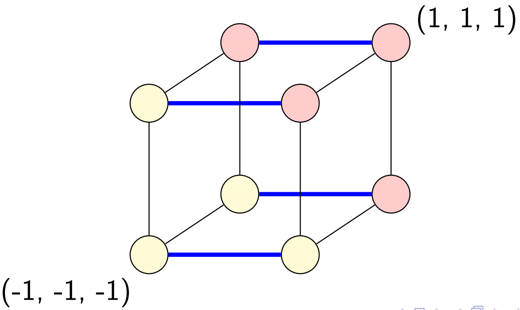
- Influence at coordinate i, $\mathbf{Inf}_i$: prob. that voter i changes outcome.
- Influence of f: $\mathbf{I}[f] = \sum_{i=1}^{n} \mathbf{Inf}_i[f]$.
- Example: $\mathbf{I}[\mathrm{Maj}_3(x)] = 3/2$.



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- Nassau County (NY) voting system:

$$f(x) = \mathrm{sgn}(-58 + 31x_1 + 31x_2 + 28x_3 + 21x_4 + 2x_5 + 2x_6).$$

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- Lawyer Banzhaf sued Nassau County board (1965).

Influences of Tribes, Majority

f monotone: $x \leq y$ coordinate-wise $\Rightarrow f(x) \leq f(y)$.

Theorem

$\mathbf{I}[f] \leq \mathbf{I}[Maj_n] = \sqrt{2/\pi}\sqrt{n} + O(n^{-1/2})$ *for all monotone f.*

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- For $n = ws$, define $\text{Tribes}_n = \text{Tribes}_{w,s}$ with w, s such that $\text{Tribes}_{w,s}$ is essentially unbiased.
- $\mathbf{Inf}_i[\text{Tribes}_n] = \frac{\ln n}{n} \cdot (1 + o(1))$.

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Theorem (Kahn, Kalai, Linial)

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- Application: bribing voters.

Noise Stability

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Definition

For a fixed $x \in \{-1, 1\}^n$ and $\rho \in [0, 1]$, $\mathbf{y}$ is ρ-correlated with x if, for each coordinate i,

$$\mathbf{y}_i = \begin{cases} x_i & \text{with probability } \rho \\ \text{randomly chosen} & \text{with probability } 1 - \rho \end{cases}.$$

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Definition

For a Boolean function f and $\rho \in [0, 1]$, the noise stability of f at ρ is

$$\mathbf{Stab}_\rho[f] = E[f(\mathbf{x})f(\mathbf{y})].$$

for $\mathbf{x}$ uniformly random and $\mathbf{y}$ ρ-correlated with $\mathbf{x}$.

Noise Stability: Voting Example

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Imagine:

- Voting system represented by f.
- Some chance $\frac{1-\rho}{2}$ that the vote is misrecorded.
- Noise stability: measure of how much f is resistant to misrecorded votes.

The Noise Stability of Majority

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Theorem (Majority is Stablest)

Among Boolean functions that are unbiased and have only small influences, the Majority function has approximately the largest noise stability.

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- The Condorcet winner is the candidate that wins all his/her elections.
- May not always occur: might be some situations in which each candidate loses a pairwise election.
- Goal: find a function in which this contradiction never occurs.

Example: Contradiction with f Majority

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- A wins the pairwise election with B.
- C wins the pairwise election with A.
- B wins the pairwise election with C.
- There is no Condorcet winner!

Arrow's Theorem: The Statement

Theorem (Arrow's Theorem)

In an n-candidate Condorcet election, if there is always a Condorcet winner, then $f(x) = \pm x_i$ *for some* i *(dictatorship).*

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In a 3-candidate Condorcet election, the probability of a Condorcet winner is exactly $3/4(1 - \mathbf{Stab}_{-1/3}[f])$.

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Dictator: only function for which $\mathbf{Stab}_{-1/3}[f] = -1/3 \Rightarrow$
$\frac{3}{4}(1 - \mathbf{Stab}_{-1/3}[f]) = 1$.

Peres's Theorem

- Noise sensitivity of f at δ is *probability* that misrecorded votes *change* outcome:

$$\mathbf{NS}_\delta[f] = \frac{1}{2} - \frac{1}{2}\mathbf{Stab}_{1-2\delta}[f].$$

Theorem (Peres, 1999)

For any LTF f, $\mathbf{NS}_\delta[f] \leq O(\sqrt{\delta})$.

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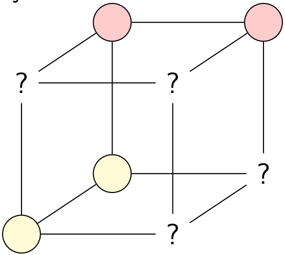
Applications of Peres's theorem

Application: learning theory.



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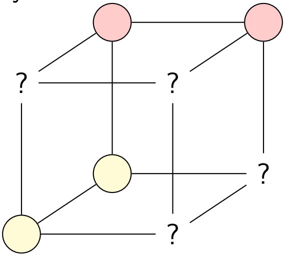

Corollary

An AND of 2 LTFs is learnable with error ϵ *in time* $n^{O(1/\epsilon^2)}$.

Open problem: extend Peres's theorem to polynomial threshold functions: $\mathrm{sgn}(p(x))$.

Fourier Analysis of Boolean Functions

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For example, $\max_2(x_1, x_2)$, outputs the maximum of x_1 and x_2:

$$\max_2(x_1, x_2) = \frac{1}{2} + \frac{1}{2}x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_1x_2.$$

Uniqueness of the Fourier Expansion

For a given f: always exists a Fourier expansion. In particular:

Theorem

Every Boolean function can be uniquely *expressed as a multilinear polynomial, called its Fourier expansion,*

$$f(x) = \sum_{S \subseteq [n]} \hat{f}(S) x^S,$$

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Coefficients $\hat{f}(S)$: Fourier spectrum of f.

Parseval's and Plancherel's Theorems

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For any Boolean functions f and g,

$$\mathbf{E}[f(\mathbf{x})g(\mathbf{x})] = \sum_{S \subseteq [n]} \hat{f}(S)\hat{g}(S).$$

Applies equally well to real-valued functions. Also yields corollary:

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Theorem (Parseval)

For any Boolean function f,

$$\sum_{S \subseteq [n]} \hat{f}(S)^2 = E[f(\mathbf{x})^2] = 1.$$

Fourier Expansions for Stability, Influence

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$$\mathbf{Stab}_\rho[f] = \sum_{S \subseteq [n]} \rho^{|S|} \hat{f}(S)^2.$$

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Acknowledgements

We would like to thank:

- Yufei Zhao
- MIT-PRIMES
- Our parents